

SHORT CONJUGATORS AND COMPRESSION EXPONENTS IN FREE SOLVABLE GROUPS

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ABSTRACT. Free solvable groups have often been studied using the Magnus embedding which, with the aid of Fox calculus, we show is a quasi-isometric embedding. In particular, we use this result to obtain a polynomial upper bound for the length of short conjugators in free solvable groups. To do this, we also need to study the same problem in wreath products in general. We also use the Magnus embedding to obtain a non-zero lower bound on L_p compression exponents in free solvable groups.

1. INTRODUCTION

The Magnus embedding is a valuable tool in the study of free solvable groups. If N is a normal subgroup of a (non-abelian) free group F of rank r , whose derived subgroup is denoted N' , then the Magnus embedding expresses F/N' as a subgroup of the wreath product $\mathbb{Z}^r \wr F/N$. The embedding was introduced in 1939 by Wilhelm Magnus [Mag39], but in the 1950's Fox, with a series of papers [Fox53], [Fox54], [Fox56], [CFL58], [Fox60], developed a notion of calculus on free groups which enabled the Magnus embedding to be further exploited. We will show the following:

Theorem 1. *The Magnus embedding $\varphi : F/N' \hookrightarrow \mathbb{Z}^r \wr F/N$ is a quasi-isometric embedding.*

The main motivation for this paper was to study the length of conjugators in free solvable groups. This is a notion related to the conjugacy problem, one of Max Dehn's three decision problems for groups given in 1911 [Deh11]. Let Γ be a finitely presented group with finite generating set X . The conjugacy problem asks whether there is an algorithm which determines when two given words on $X \cup X^{-1}$ represent conjugate elements in Γ . This question may also be asked of recursively presented groups, and we can try to develop our understanding further by asking whether one can find, in some sense, a short conjugator between two given conjugate elements of a group.

Suppose word-lengths in Γ , with respect to the given generating set X , are denoted by $|\cdot|$. The *conjugacy length function* was introduced by T. Riley and is the minimal function $\text{CLF}_\Gamma : \mathbb{N} \rightarrow \mathbb{N}$ which satisfies the following: if u is conjugate to v in Γ and $|u| + |v| \leq n$ then there exists a conjugator $\gamma \in \Gamma$ such that $|\gamma| \leq \text{CLF}_\Gamma(n)$. One can define it more concretely to be the function which sends an integer n to

$$\max\{\min\{|w| : wu = vw\} : |u| + |v| \leq n \text{ and } u \text{ is conjugate to } v \text{ in } \Gamma\}.$$

We know various upper bounds for the conjugacy length function in certain classes of groups. For example, Gromov-hyperbolic groups have a linear upper bound; this is demonstrated by Bridson and Haefliger [BH99, Ch.III.Γ Lemma 2.9]. They also show that $\text{CAT}(0)$ groups have an exponential upper bound for conjugacy length [BH99, Ch.III.Γ Theorem 1.12]. In [Sal11] we showed that some metabelian groups, namely the lamplighter groups, solvable Baumslag-Solitar groups and lattices in

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SOL, have a linear upper bound on conjugacy length. A consequence of the result for lattices in SOL, together with results of Behrstock and Druţu [BD11] and Ji, Ogle and Ramsey [JOR10], was that fundamental groups of prime 3-manifolds have a quadratic upper bound. Behrstock and Druţu also show that mapping class groups have a linear upper bound, expanding on results of Masur and Minsky [MM00] in the pseudo-Anosov case and Jing Tao [Tao11] for the reducible case.

In this paper we look at wreath products and free solvable groups. The definition of a free solvable group is as follows: let $F' = [F, F]$ denote the derived subgroup of F , where F is the free group of rank r . Denote by $F^{(d)}$ the d -th derived subgroup, that is $F^{(d)} = [F^{(d-1)}, F^{(d-1)}]$. The *free solvable group* of rank r and derived length d is the quotient $S_{r,d} = F/F^{(d)}$. The conjugacy problem in free solvable groups was shown, using the Magnus embedding, to be solvable by Remeslennikov and Sokolov [RS70], extending the same result for free metabelian groups by Matthews [Mat66]. Recently, Vassileva [Vas11] has looked at the computational complexity of algorithms to solve the conjugacy problem and the conjugacy search problem in wreath products and free solvable groups. In particular Vassileva showed that the complexity of the conjugacy search problem in free solvable groups is at most polynomial of degree 8. Using Theorem 1 we are able to improve our understanding of the length of short conjugators in free solvable groups:

Theorem 2. *Let $d \geq 1$ and $r > 1$. Then the conjugacy length function of the free solvable group $S_{r,d}$ is bounded above by a polynomial of degree at most $2^{d+1} - 1$.*

In order to apply the Magnus embedding, we need to understand conjugacy in wreath products. For such groups the conjugacy problem was studied by Matthews [Mat66], who showed that for two recursively presented groups A, B with solvable conjugacy problem, their wreath product $A \wr B$ has solvable conjugacy problem if and only if B has solvable power problem. In Section 3 we show that for such A and B there is an upper bound for the conjugacy length function of $A \wr B$ which depends on the conjugacy length functions of A and B and on the subgroup distortion of infinite cyclic subgroups of B . In the case that B is torsion-free (or at least the case when the two conjugate elements under consideration are of infinite order) the conjugacy length does not depend on the conjugacy length in A .

Theorem 3. *Suppose A and B are recursively presented groups with solvable conjugacy problem such that B also has solvable power problem. Let $u = (f, b), v = (g, c)$ be elements in $\Gamma = A \wr B$. Then u, v are conjugate if and only if there exists a conjugator $\gamma \in \Gamma$ such that*

$$\begin{aligned} d_\Gamma(1, \gamma) &\leq (n+1)P(2\delta_{(b)}^B(P) + 1) && \text{if } b \text{ is of infinite order, or} \\ d_\Gamma(1, \gamma) &\leq P(N+1)(2n + \text{CLF}_A(n) + 1) && \text{if } b \text{ is of finite order } N, \end{aligned}$$

where $n = d_\Gamma(1, u) + d_\Gamma(1, v)$, δ_H^B is the subgroup distortion function of $H < B$ and $P = 7n$ if (f, b) is not conjugate to $(1, b)$ and $P = n + 2\text{CLF}_B(n)$ otherwise.

In order to apply Theorem 3 to free solvable groups we must understand the distortion of their cyclic subgroups. Given a finitely generated subgroup H in a finitely generated group G , with corresponding word metrics d_H and d_G respectively, the *subgroup distortion function* δ_H^G compares the size of an element in a Cayley graph of H with its size in a Cayley graph of G . It is defined as

$$\delta_H^G(n) = \max\{d_H(e_G, h) \mid d_G(e_G, h) \leq n\}.$$

Subgroup distortion is studied up to an equivalence relation of functions. For functions $f, g : \mathbb{N} \rightarrow [0, \infty)$ we write $f \preceq g$ if there exists an integer $C > 0$ such that $f(n) \leq Cg(Cn)$ for all $n \in \mathbb{N}$. The two functions are equivalent if both $f \preceq g$ and $g \preceq f$. In this case we write $f \asymp g$. Up to this equivalence we can talk about *the*

distortion function for a group. If the distortion function of a subgroup H satisfies $\delta_H^G \asymp n$ then we say H is undistorted in G , otherwise H is distorted.

The structure of the paper is as follows: we begin in Section 2 with the appropriate preliminary definitions, including a brief account of Fox calculus. This section builds up to Theorem 1. In Section 3 we study conjugacy in wreath products, ultimately obtaining Theorem 3. The subgroup distortion of cyclic subgroups of $S_{r,d}$ is discussed in Section 4, before we move onto conjugacy in free solvable groups in the penultimate section.

Finally, in Section 6, we apply Theorem 1 to study the L_p compression exponents for free solvable groups. Compression exponents were first introduced by Guentner and Kaminker [GK04], building on the idea of uniform embeddings introduced by Gromov [Gro93]. In particular we show that free solvable groups have non-zero Hilbert compression exponent.

Theorem 4. *For $r, d \in \mathbb{N}$, $r, d \neq 1$, the L_p compression exponent for $S_{r,d}$ satisfies*

$$\frac{1}{d-1} \min \left\{ \frac{1}{p}, \frac{1}{2} \right\} \leq \alpha_p^*(S_{r,d}).$$

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2. PRELIMINARIES

2.1. Restricted Wreath Product. Let A, B be groups. Denote by $A^{(B)}$ the set of all functions from B to A with finite support, and equip it with pointwise multiplication to make it a group. The (restricted) wreath product $A \wr B$ is the semidirect product $A^{(B)} \rtimes B$. To be more precise, the elements of $A \wr B$ are pairs (f, b) where $f \in A^{(B)}$ and $b \in B$. Multiplication in $A \wr B$ is given by

$$(f, b)(g, c) = (fg^b, bc), \quad f, g \in A^{(B)}, \quad b, c \in B$$

where $g^b(x) = g(b^{-1}x)$ for each $x \in B$. The identity element in B will be denoted by e_B , while we use 1 to denote the trivial function from B to A .

The following Lemma deals with the word-length of elements in $\Gamma = A \wr B$ when A, B are finitely generated. It was given by de Cornulier (whose proof we follow here) in the Appendix of [dC06] in a slightly more general context, and also by Davis and Olshanskii [DO11, Theorem 3.4]. We fix a generating set X for B , let $S = X \cup X^{-1}$, and for each $b \in B$ denote the corresponding word-length as $|b|$. We consider the left-invariant word metric on B , given by $d_B(x, y) := |x^{-1}y|$. Similarly, fix a finite generating set T for A and let $|\cdot|$ denote the word-length. For $f \in A^{(B)}$, let

$$|f| = \sum_{x \in B} |f(x)|.$$

Let A_{e_B} be the subgroup of $A^{(B)}$ consisting of those elements whose support is contained in $\{e_B\}$. Then A_{e_B} is generated by $\{f_t \mid t \in T\}$ where $f_t(e_B) = t$ for each $t \in T$ and Γ is generated by $\{(1, s), (f_t, e_B) \mid s \in S, t \in T\}$. With respect to this generating set, let $|(f, b)|$ denote the corresponding word-length for $(f, b) \in \Gamma$.

Lemma 2.1 (de Cornulier [dC06]). *Let $(f, b) \in \Gamma = A \wr B$, where A, B are finitely generated groups. Then*

$$|(f, b)| = K(\text{Supp}(f), b) + |f|$$

where $K(\text{Supp}(f), b)$ is the shortest path in the Cayley graph $\text{Cay}(B, S)$ of B from e_B to b , travelling via every point in $\text{Supp}(f)$.

Proof. Let $n = K(\text{Supp}(f), b)$, and suppose $e_B = c_0, c_1, \dots, c_n = b$ are the vertices of a path in the Cayley graph of B such that if x is in the support of f then $x = c_i$ for some $i \in \{0, 1, \dots, n\}$. For each i , let s_i be the element in S so that $c_{i+1} = c_i s_i$ and let $f_i \in A_{e_B}$ be the function such that $f_i(e_B) = f(c_i)$ unless $c_i = c_j$ for some $j < i$, in which case $f_i = 1$. Then

$$\begin{aligned} (f, b) &= (f_0 f_1^{c_1} \dots f_n^{c_n}, b) \\ &= (f_0, s_1)(f_1, s_2) \dots (f_{n-1}, s_n)(f_n, e_B) \\ &= (f_0, e_B)(1, s_1)(f_1, e_B)(1, s_2) \dots (1, s_n)(f_n, e_B). \end{aligned}$$

For each i , the element (f_i, e_B) has length equal to $|f_i(e_B)|$, hence we have the following upper bound:

$$\begin{aligned} |(f, b)| &\leq n + \sum_{i=1}^n |f_i(e_B)| \\ &= K(\text{Supp}(f), b) + |f|. \end{aligned}$$

Now suppose that we can express (f, b) as the product of k generators. After clustering the adjacent generators of the form (f_i, e_B) we can write

$$(f, b) = (f_0, e_B)(1, s_1)(f_1, e_B)(1, s_2) \dots (f_{m-1}, e_B)(1, s_m)(f_m, e_B)$$

for some integer m . It follows from this expression that if $c_i = s_1 \dots s_i$ then $b = c_m$ and $f = f_0 f_1^{c_1} \dots f_m^{c_m}$. Since each f_i is in A_{e_B} the support of f is therefore contained in the set $\{e_B, c_1, c_2, \dots, c_m = b\}$. Thus $e_B, c_1, c_2, \dots, c_m$ describes a path in the Cayley graph of B of length m starting at e_B , passing through every point in $\text{Supp}(f)$ and finishing at b . Subsequently $m \geq K(\text{Supp}(f), b)$. Finally, note that each (f_i, e_B) has word-length equal to $|f_i(e_B)|$. Hence

$$k = m + \sum_{i=1}^m |f_i(e_B)| = m + |f| \geq K(\text{Supp}(f), b) + |f|$$

and the Lemma follows. \square

2.2. Fox Calculus. In Section 2.3 we will introduce the Magnus embedding. This is a tool that helps us study the free solvable groups $S_{r,d}$. In order to make effective use of the Magnus embedding we need to understand Fox derivatives. These were introduced by Fox in the 1950's in a series of papers [Fox53], [Fox54], [Fox56], [CFL58], [Fox60].

Recall that a derivation on a group ring $\mathbb{Z}(G)$ is a mapping $\mathcal{D} : \mathbb{Z}(G) \rightarrow \mathbb{Z}(G)$ which satisfies the following two conditions for every $a, b \in \mathbb{Z}(G)$:

$$\begin{aligned} \mathcal{D}(a + b) &= \mathcal{D}(a) + \mathcal{D}(b) \\ \mathcal{D}(ab) &= \mathcal{D}(a)\varepsilon(b) + a\mathcal{D}(b) \end{aligned}$$

where $\varepsilon : \mathbb{Z}(G) \rightarrow \mathbb{Z}$ sends each element of G to 1. That is, for $g_1, \dots, g_n \in G$ and $\alpha_1, \dots, \alpha_n$ integers, $\varepsilon(\alpha_1 g_1 + \dots + \alpha_n g_n) = \alpha_1 + \dots + \alpha_n$.

Suppose $G = F$, the free group on generators $X = \{x_1, \dots, x_r\}$. For each generator we can define a unique derivation $\frac{\partial}{\partial x_i}$ which satisfies

$$\frac{\partial x_j}{\partial x_i} = \delta_{ij}$$

where δ_{ij} is the Kronecker delta. Any derivation \mathcal{D} can be expressed as a linear combination of these: for each \mathcal{D} there exists some elements $k_i \in \mathbb{Z}(F)$ such that

$$\mathcal{D}(a) = \sum_{i=1}^n k_i \frac{\partial a}{\partial x_i}$$

for each $a \in \mathbb{Z}(F)$.

Fox describes the following Lemma as the “fundamental formula” and it can be found in [Fox53, (2.3)].

Lemma 2.2 (Fundamental formula of Fox calculus). *Let $a \in \mathbb{Z}(F)$. Then*

$$a - \varepsilon(a)1 = \sum_{i=1}^r \frac{\partial a}{\partial x_i} (x_i - 1).$$

Fox derivatives also accept a form of integration, see [CF63, Ch.VII (2.10)]. In particular, given $\beta_1, \dots, \beta_r \in \mathbb{Z}(F)$ one can find $c \in \mathbb{Z}(F)$ such that $\frac{\partial c}{\partial x_i} = \beta_i$ for each i . The element c is unique up to addition of scalar multiples of the identity.

Given a normal subgroup N in F and a derivation \mathcal{D} of $\mathbb{Z}(F)$ we can define a derivation $\mathcal{D}^* : \mathbb{Z}(F) \rightarrow \mathbb{Z}(F/N)$ through the composition of maps

$$\begin{array}{ccc} \mathbb{Z}(F) & \xrightarrow{\mathcal{D}} & \mathbb{Z}(F) \\ & \searrow \mathcal{D}^* & \downarrow \alpha^* \\ & & \mathbb{Z}(F/N) \end{array}$$

where α^* is the extension of the quotient homomorphism $\alpha : F \rightarrow F/N$.

The following Lemma can be deduced from the Magnus embedding, but it also follows from [Fox53, (4.9)].

Lemma 2.3. *Let $g \in F$. Then $\mathcal{D}^*(g) = 0$ for every derivation \mathcal{D} if and only if $g \in N' = [N, N]$.*

Strictly speaking, α^* is a map from $\mathbb{Z}(F)$ to $\mathbb{Z}(F/N)$, but if instead we consider the canonical map $\alpha : F/N' \rightarrow F/N$ then α^* becomes a map from $\mathbb{Z}(F/N')$ to $\mathbb{Z}(F/N)$. Understanding the kernel of α^* will be helpful in Section 5.

Lemma 2.4 (See also Gruenberg [Gru67, §3.1 Theorem 1]). *An element of the kernel of $\alpha^* : \mathbb{Z}(F/N') \rightarrow \mathbb{Z}(F/N)$ can be written in the form*

$$\sum_{j=1}^m r_j (h_j - 1)$$

for some integer m , where $r_j \in F/N'$ and $h_j \in N/N'$ for each $j = 1, \dots, m$.

Proof. Take an arbitrary element a in the kernel of α^* . Suppose we can write

$$a = \sum_{g \in F/N'} \beta_g g$$

where $\beta_g \in \mathbb{Z}$ for each $g \in F/N'$. Fix a coset xN . Then

$$\sum_{\alpha(g)=xN} \beta_g = 0$$

since this is the coefficient of xN in $\alpha^*(a)$. Notice that $\alpha(g) = xN$ if and only if there is some $h \in N$ such that $g = xh$. Thus the sum can be rewritten as

$$\sum_{h \in N \setminus \{1\}} \beta_{xh} = -\beta_x.$$

This leads us to

$$\sum_{\alpha(g)=xN} \beta_g g = \sum_{h \in N \setminus \{1\}} \beta_{xh} x(h-1)$$

which implies the Lemma after summing over all left-cosets. \square

2.3. The Magnus Embedding. Let F be the free group of rank r with generators $X = \{x_1, \dots, x_r\}$ and let N be a normal subgroup of F . The Magnus embedding gives a way of recognising F/N' , where N' is the derived subgroup of N , as a subgroup of the wreath product $M(F/N) = \mathbb{Z}^r \wr F/N$.

Consider the group ring $\mathbb{Z}(F/N)$ and let \mathcal{R} be the free $\mathbb{Z}(F/N)$ -module with generators t_1, \dots, t_r . We define a homomorphism

$$\varphi : F \longrightarrow M(F/N) = \begin{pmatrix} F/N & \mathcal{R} \\ 0 & 1 \end{pmatrix} = \left\{ \begin{pmatrix} g & a \\ 0 & 1 \end{pmatrix} \mid g \in F/N, a \in \mathcal{R} \right\}$$

by

$$\varphi(w) = \begin{pmatrix} \alpha(w) & \frac{\partial^* w}{\partial x_1} t_1 + \dots + \frac{\partial^* w}{\partial x_r} t_r \\ 0 & 1 \end{pmatrix}$$

where α is the quotient homomorphism $\alpha : F \rightarrow F/N$. Magnus [Mag39] recognised that the kernel of φ is equal to N' and hence φ induces an injective homomorphism from F/N' to $M(F/N)$ which is known as the *Magnus embedding*. In the rest of this paper we will use φ to denote both the homomorphism defined above and the Magnus embedding it induces.

Given $w \in F$, its image under the Magnus embedding can be identified with $(f, b) \in \mathbb{Z}^r \wr F/N$ in the following way: we take $b = \alpha(w)$ and f will be the function $f^w = (f_1^w, \dots, f_r^w)$, where for each i the function $f_i^w : F/N \rightarrow \mathbb{Z}$ satisfies the equation

$$\sum_{g \in F/N} f_i^w(g) g = \frac{\partial^* w}{\partial x_i} \in \mathbb{Z}(F/N).$$

Let $d_{F/N'}$ denote the word metric in F/N' with respect to the generators determined by the elements of X and let d_M denote the word metric on $M(F/N)$ with respect to the generating set

$$\left\{ \begin{pmatrix} \alpha(x_1) & 0 \\ 0 & 1 \end{pmatrix}, \dots, \begin{pmatrix} \alpha(x_r) & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & t_1 \\ 0 & 1 \end{pmatrix}, \dots, \begin{pmatrix} 1 & t_r \\ 0 & 1 \end{pmatrix} \right\}.$$

Note that this generating set is the same as that used for Lemma 2.1. The aim is to compare the metrics $d_{F/N'}$ and d_M . We first give a result of Droms, Lewin and Servatius [DLS93, Theorem 2], which is also observed by Myasnikov, Roman'kov, Ushakov and Vershik [MRUV10, Theorem 2.11], on the word-length of elements in F/N' . In order to do so we need to set up some notation.

Let E be the edge set of the Cayley graph $\text{Cay}(F/N, X)$ of F/N with respect to the generating set determined by X . Given a word w on X we obtain a path ρ_w in $\text{Cay}(F/N, X)$. Define a function $\pi_w : E \rightarrow \mathbb{Z}$ such that for each edge $(g, gx) \in E$ the value of $\pi_w(g, gx)$ is equal to the net number of times the path ρ_w traverses this edge — for each time the path travels from g to gx count $+1$; for each time the path goes from gx to g count -1 . Since the path is finite, π_w has finite support.

Let $\text{Supp}(\pi_w)$ denote the subgraph of $\text{Cay}(F/N, X)$ containing all edges e such that $\pi_w(e) \neq 0$. Consider a new path $\sigma(\pi_w)$ which is a path travelling through every point in $\text{Supp}(\pi_w) \cup \{1\}$ so that it minimises the number of edges not contained in $\text{Supp}(\pi_w)$. Let $W(\pi_w)$ denote this number.

Lemma 2.5 (Droms–Lewin–Servatius [DLS93]). *Let w be a word on generators X which determines the element $g \in F/N'$. Then*

$$d_{F/N'}(1, g) = \sum_{e \in E} |\pi_w(e)| + 2W(\pi_w).$$

Theorem 2.6. *The subgroup $\varphi(F/N')$ is undistorted in $M(F/N)$. To be precise, for each $g \in F/N'$*

$$\frac{1}{2}d_{F/N'}(1, g) \leq d_M(1, \varphi(g)) \leq 2d_{F/N'}(1, g).$$

Proof. The aim is to compare the word-lengths given by Lemma 2.5 and Lemma 2.1. Let w be a word on X representing $g \in F/N$. The image of g under the Magnus embedding is $\varphi(w) = (f^w, \alpha(w))$, with $f^w = (f_1^w, \dots, f_r^w)$ satisfying

$$\sum_{g \in F/N} f_i^w(g)g = \frac{\partial^* w}{\partial x_i} \in \mathbb{Z}(F/N).$$

We claim that $f_i^w(g) = \pi_w(g, gx_i)$, and will prove this by induction on the word-length of w (see also [MRUV10, Lemma 2.6]). If $w = x_j$ then $\frac{\partial^* w}{\partial x_i} = \delta_{ij}$. The path ρ_w consists of just one edge: $(1, x_j)$. Hence $\pi_w(g, gx_i)$ is zero everywhere except when $g = 1$ and $i = j$, where it takes the value 1. Thus, in this case, the claim holds. If $w = x_j^{-1}$ then $\frac{\partial^* w}{\partial x_i} = -\delta_{ij}x_j^{-1}$. The path ρ_w this time consists of the edge $(x_j^{-1}, 1)$ and one can check that the claim holds here too.

Now suppose w has length at least 2 and that the claim holds for all words shorter than w . Suppose also that w is of the form $w = w'x_j^\varepsilon$ where w' is a non-trivial word and $\varepsilon = \pm 1$. Then

$$\frac{\partial^*(w'x_j^\varepsilon)}{\partial x_i} = \frac{\partial^* w'}{\partial x_i} + \alpha(w') \frac{\partial^* x_j^\varepsilon}{\partial x_i}$$

and it follows that $f_i^{w'}(g) = f_i^{w'x_j^\varepsilon}(g)$ whenever $i \neq j$. When $i = j$ we get

$$f_i^w(g) = \begin{cases} f_i^{w'}(g) & \text{if } \varepsilon = 1 \text{ and } g \neq \alpha(w'), \text{ or } \varepsilon = -1 \text{ and } g \neq \alpha(w), \\ f_i^{w'}(g) + 1 & \text{if } \varepsilon = 1 \text{ and } g = \alpha(w'), \\ f_i^{w'}(g) - 1 & \text{if } \varepsilon = -1 \text{ and } g = \alpha(w). \end{cases}$$

Meanwhile, ρ_w is obtained from $\rho_{w'}$ by attaching one extra edge on to its final vertex, namely the edge $(w', w'x_i)$ if $\varepsilon = 1$ or (w, wx_i) if $\varepsilon = -1$. Hence $\pi_w(g, gx_i) = \pi_{w'}(g, gx_i)$ whenever $i \neq j$ and when $i = j$ we get

$$\pi_w(g, gx_i) = \begin{cases} \pi_{w'}(g, gx_i) & \text{if } \varepsilon = 1 \text{ and } g \neq \alpha(w'), \text{ or } \varepsilon = -1 \text{ and } g \neq \alpha(w) \\ \pi_{w'}(g, gx_i) + 1 & \text{if } \varepsilon = 1 \text{ and } g = \alpha(w'), \\ \pi_{w'}(g, gx_i) - 1 & \text{if } \varepsilon = -1 \text{ and } g = \alpha(w). \end{cases}$$

Thus, applying the inductive hypothesis gives $f_i^w(g) = \pi_w(g, gx_i)$ and the claim therefore holds for all words w .

From Lemma 2.1 the word-length in $M(F/N)$ is given by

$$d_M(1, (f^w, \alpha(w))) = K(\text{Supp}(f^w), \alpha(w)) + \sum_{y \in F/N} \|f^w(y)\|$$

where $\|\cdot\|$ is the ℓ_1 -norm on \mathbb{Z}^r . The above expression of f^w in terms of π_w leads us to the equation

$$\sum_{y \in F/N} \|f^w(y)\| = \sum_{e \in E} |\pi_w(e)|.$$

Since an edge e is in $\text{Supp}(\pi_w)$ only if one of its ends is in $\text{Supp}(f^w)$, we see that $\text{Supp}(f^w)$ is contained in the subgraph $\text{Supp}(\pi_w)$. Take a path q starting at 1 and travelling through every point in $\text{Supp}(f^w)$, in particular we may take q to be a path realising $K(\text{Supp}(f^w), \alpha(w))$. Any edge in $\text{Supp}(\pi_w)$ which is not in this path

must have one vertex lying in the path q . Adding these edges to q (along with the corresponding backtracking) gives a new path q' passing through every point of $\text{Supp}(\pi_w) \cup \{1\}$. Note that every edge in q' that is not in $\text{Supp}(\pi_w)$ was already in q . Hence the length of q is bounded below by the size of $W(\pi_w)$. In particular $W(\pi_w) \leq K(\text{Supp}(f^w), \alpha(w))$ and hence

$$\frac{1}{2}d_{F/N'}(1, g) \leq d_M(1, \varphi(g)).$$

On the other hand, suppose $w = x_{i_1}^{\varepsilon_1} \dots x_{i_m}^{\varepsilon_m}$ is a minimal word representing g , that is $d_{F/N'}(1, g) = m$. Then $\varphi(g) = \varphi(x_{i_1})^{\varepsilon_1} \dots \varphi(x_{i_m})^{\varepsilon_m}$ gives an expression for $\varphi(g)$ in terms of $2m$ generators, since for each i_j

$$\begin{aligned} \varphi(x_{i_j}) &= \begin{pmatrix} x_{i_j} & t_{i_j} \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & t_{i_j} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_{i_j} & 0 \\ 0 & 1 \end{pmatrix}. \end{aligned}$$

Hence $d_M(1, \varphi(g)) \leq 2m$ and the result follows. \square

3. CONJUGACY IN WREATH PRODUCTS

Let A, B be finitely generated, recursively presented groups such that both A and B have solvable conjugacy problem and B has solvable power problem. Let $\Gamma = A \wr B$. By a result of Matthews [Mat66], Γ has solvable conjugacy problem. We retain these hypotheses on A and B for the rest of this section.

Fix $b \in B$ and let $\{t_i \mid i \in I\}$ be a set of right-coset representatives for $\langle b \rangle$ in B . We associate to this a family of maps $\pi_{t_i}^{(z)} : A^{(B)} \rightarrow A$ for each z in B as follows:

$$\pi_{t_i}^{(z)}(f) = \begin{cases} \prod_{j=0}^{N-1} f(z^{-1}b^j t_i) & \text{for } b \text{ of finite order } N \\ \prod_{j=-\infty}^{\infty} f(z^{-1}b^j t_i) & \text{for } b \text{ of infinite order.} \end{cases}$$

The products above are taken so that the order of multiplication is such that $f(t_i b^j z^{-1})$ is to the left of $f(t_i b^{j-1} z^{-1})$ for each j . When $z = e_B$ we denote $\pi_{t_i}^{(z)}$ by π_{t_i} .

Proposition 3.1 (Matthews [Mat66]). *Two elements (f, b) and (g, c) are conjugate in $A \wr B$ if and only if there exists an element z in B such that $bz = zc$ and for all $i \in I$ either*

- $\pi_{t_i}^{(z)}(g) = \pi_{t_i}(f)$ if b is of infinite order; or
- $\pi_{t_i}^{(z)}(g)$ is conjugate to $\pi_{t_i}(f)$ if b is of finite order.

For such z in B , a corresponding function h such that $(f, b)(h, z) = (h, z)(g, c)$ is defined as follows: if b is of infinite order then for each $i \in I$ and each $k \in \mathbb{Z}$ we set

$$h(b^k t_i) = \left(\prod_{j \leq k} f(b^j t_i) \right) \left(\prod_{j \leq k} g(z^{-1}b^j t_i) \right)^{-1}$$

or if b is of finite order N , then for each $i \in I$ and each $k = 0, \dots, N-1$ we set

$$h(b^k t_i) = \left(\prod_{j=0}^k f(b^j t_i) \right) \alpha_{t_i} \left(\prod_{j=0}^k g(z^{-1}b^j t_i) \right)^{-1}$$

where α_i satisfies $\pi_{t_i}(f)\alpha_{t_i} = \alpha_{t_i}\pi_{t_i}^{(z)}(g)$.

3.1. Short Conjugators. Proposition 3.1 gives us an explicit description of a particular conjugator for two elements in $A \wr B$. The following Lemma tells us that any conjugator between two elements has a concrete description similar to that given by Matthews in the preceding Proposition. With this description at our disposal we will be able to determine their size and thus find a short conjugator.

Lemma 3.2. *Let $(h, z), (f, b), (g, c) \in A \wr B$ be such that $(f, b)(h, z) = (h, z)(g, c)$. Then there is a set of right-coset representatives $\{t_i \mid i \in I\}$ of $\langle b \rangle$ in B such that, if b is of infinite order then*

$$h(b^k t_i) = \left(\prod_{j \leq k} f(b^j t_i) \right) \left(\prod_{j \leq k} g(z^{-1} b^j t_i) \right)^{-1}$$

for every $i \in I$ and $k \in \mathbb{Z}$. If b is of finite order N then

$$h(b^k t_i) = \left(\prod_{j=0}^k f(b^j t_i) \right) \alpha_{t_i} \left(\prod_{j=0}^k g(z^{-1} b^j t_i) \right)^{-1}$$

for every $i \in I$ and $k = 0, \dots, N-1$ and where α_{t_i} is any element satisfying $\pi_{t_i}(f) \alpha_{t_i} = \alpha_{t_i} \pi_{t_i}^{(z)}(g)$.

Proof. Fix a set of coset representatives $\{s_i \mid i \in I\}$. By Matthews' result there exists a conjugator $(h_1, z_1) \in A \wr B$ for (f, b) and (g, c) as described in Proposition 3.1, with respect to the coset representatives $\{s_i \mid i \in I\}$. Since (h, z) and (h_1, z_1) are both conjugators then it follows that there exists some (ψ, y) in $Z_\Gamma(f, b)$ such that $(h, z) = (\psi, y)(h_1, z_1)$. This tells us that $z = yz_1$ and also that $h(x) = \psi(x)h_1(y^{-1}x)$ for each $x \in B$. Since (ψ, y) is in the centraliser of (f, b) , we obtain two identities:

- (1) $yb = by$
- (2) $\psi(x)f(y^{-1}x) = f(x)\psi(b^{-1}x) \quad \forall x \in B.$

For each $i \in I$ we set $t_i = ys_i$. Then

$$\begin{aligned} h(b^k t_i) &= \psi(b^k t_i) h_1(y^{-1} b^k t_i) \\ &= \psi(b^k t_i) h_1(b^k s_i) \\ &= \psi(b^k t_i) \left(\prod_{j \leq k} f(b^j s_i) \right) \left(\prod_{j \leq k} g(z_1^{-1} b^j s_i) \right)^{-1} \\ &= \psi(b^k t_i) \left(\prod_{j \leq k} f(y^{-1} b^j t_i) \right) \left(\prod_{j \leq k} g(z^{-1} b^j t_i) \right)^{-1}. \end{aligned}$$

We can apply equation (2) once, and then repeat this process to shuffle the ψ term past all the terms involving f . This process terminates and the ψ term vanishes because of the finiteness of support of both ψ and of f . Hence, as required, we obtain:

$$\begin{aligned} h(b^k t_i) &= f(b^k t_i) \psi(b^{k-1} t_i) \left(\prod_{j \leq k-1} f(y^{-1} b^j t_i) \right) \left(\prod_{j \leq k} g(z^{-1} b^j t_i) \right)^{-1} \\ &\vdots \\ &= \left(\prod_{j \leq k} f(b^j t_i) \right) \left(\prod_{j \leq k} g(z^{-1} b^j t_i) \right)^{-1}. \end{aligned}$$

If instead b is of finite order, N say, then for $0 \leq k \leq N-1$ we obtain

$$h(b^k t_i) = \psi(b^k t_i) \left(\prod_{j=0}^k f(y^{-1} b^j t_i) \right) \alpha_{s_i} \left(\prod_{j=0}^k g(z^{-1} b^j t_i) \right)^{-1}.$$

With equation (2) the $\psi(b^k t_i)$ term can be shuffled past the terms involving f . Unlike in the infinite order case, however, the ψ term will not vanish:

$$h(b^k t_i) = \left(\prod_{j=0}^k f(b^j t_i) \right) \psi(b^{-1} t_i) \alpha_{s_i} \left(\prod_{j=0}^k g(z^{-1} b^j t_i) \right)^{-1}.$$

All that is left to do is to verify that we may set $\alpha_{t_i} = \psi(b^{-1} t_i) \alpha_{s_i}$ and it will satisfy $\pi_{t_i}(f) \alpha_{t_i} = \alpha_{t_i} \pi_{t_i}^{(z)}(g)$:

$$\begin{aligned} e_A &= \alpha_{s_i}^{-1} \pi_{s_i}(f)^{-1} \alpha_{s_i} \pi_{s_i}^{(z_1)}(g) \\ &= \alpha_{s_i}^{-1} \left(\prod_{j=0}^{N-1} f(b^j s_i) \right)^{-1} \alpha_{s_i} \left(\prod_{j=0}^{N-1} g(z_1^{-1} b^j s_i) \right) \\ &= \alpha_{s_i}^{-1} \left(\prod_{j=0}^{N-1} f(y^{-1} b^j t_i) \right)^{-1} \alpha_{s_i} \left(\prod_{j=0}^{N-1} g(z^{-1} b^j t_i) \right) \\ &= \alpha_{s_i}^{-1} \left(\prod_{j=0}^{N-1} \psi(b^j t_i) f(b^j t_i) \psi(b^{j-1} t_i)^{-1} \right)^{-1} \alpha_{s_i} \pi_{t_i}^{(z)}(g) \\ &= \alpha_{s_i}^{-1} \psi(b^{N-1} t_i)^{-1} \pi_{t_i}(f)^{-1} \psi(b^{-1} t_i) \alpha_{s_i} \pi_{t_i}^{(z)}(g) \end{aligned}$$

and since b has order N , setting $\alpha_{t_i} = \psi(b^{-1} t_i) \alpha_{s_i}$ gives us the required form for h and the Lemma follows. \square

Obtaining a short conjugator will require two steps. Lemma 3.3 is the first of these steps. Here we actually find the short conjugator, while in Lemma 3.4 we show that the size of a conjugator (h, z) can be bounded by a function involving the size of z but independent of h altogether.

Recall that the conjugacy length function of B is the minimal function

$$\text{CLF}_B : \mathbb{N} \rightarrow \mathbb{N}$$

such that if b is conjugate to c in B and $d_B(e_B, b) + d_B(e_B, c) \leq n$ then there exists a conjugator $z \in B$ such that $d_B(e_B, z) \leq \text{CLF}_B(n)$.

Lemma 3.3. *Suppose $u = (f, b), v = (g, c)$ are conjugate elements in $\Gamma = A \wr B$ and let $n = d_\Gamma(1, u) + d_\Gamma(1, v)$. Then there exists $\gamma = (h, z) \in \Gamma$ such that $u\gamma = \gamma v$ and either:*

- (1) $d_B(e_B, z) \leq \text{CLF}_B(n)$ if (f, b) is conjugate to $(1, b)$; or
- (2) $d_B(e_B, z) \leq 3n$ if (f, b) is not conjugate to $(1, b)$.

Proof. Without loss of generality we may assume that $d_\Gamma(1, u) \leq d_\Gamma(1, v)$. By Lemma 3.2, if (h_0, z_0) is a conjugator for u and v then there exists a family of right-coset representatives $\{t_i \mid i \in I\}$ for $\langle b \rangle$ in B such that

$$\pi_{t_i}^{(z_0)}(g) = \pi_{t_i}(f) \quad \text{or} \quad \pi_{t_i}^{(z_0)}(g) \text{ is conjugate to } \pi_{t_i}(f)$$

for every $i \in I$ according to whether b is of infinite or finite order respectively (the former follows from the finiteness of the support of the function h given by Lemma 3.2).

By Proposition 3.1, (f, b) is conjugate to $(1, b)$ if and only if $\pi_{t_i}(f) = e_A$ for every $i \in I$. In this case we take

$$h(b^k t_i) = \begin{cases} \prod_{j \leq k} f(b^j t_i) & \text{if } b \text{ is of infinite order;} \\ \prod_{j=0}^k f(b^j t_i) & \text{if } b \text{ is of finite order } N \text{ and } 0 \leq k < N. \end{cases}$$

One can then verify that $(f, b)(h, e_B) = (h, e_B)(1, b)$. Thus we have reduced (1) to the case when $u = (1, b)$ and $v = (1, c)$. For this we observe that any conjugator z for b, c in B will give a conjugator $(1, z)$ for u, v in $A \wr B$. Thus (1) follows.

If on the other hand (f, b) is not conjugate to $(1, b)$ then by Proposition 3.1, $\pi_{t_i}(f) \neq e_A$ for some $i \in I$. Fix some such i , observe that there exists $k \in \mathbb{Z}$ satisfying $b^k t_i \in \text{Supp}(f)$ and there must also exist some $j \in \mathbb{Z}$ so that $z_0^{-1} b^j t_i \in \text{Supp}(g)$. Pre-multiply (h_0, z_0) by $(f, b)^{k-j}$ to get $\gamma = (h, z)$, where $z = b^{k-j} z_0$ and γ is a conjugator for u and v since $(f, b)^{k-j}$ belongs to the centraliser of u in Γ . By construction, $z^{-1} b^k t_i = z_0^{-1} b^j t_i$ and hence is contained in the support of g . From this we get an upper bound for $d_B(b^k t_i, z^{-1} b^k t_i)$ as the diameter of the set $\text{Supp}(f) \cup \text{Supp}(g)$. Note that $K(\text{Supp}(f), e_B) + K(\text{Supp}(g), e_B)$ describes the length of a path which passes through every point of $\text{Supp}(f) \cup \text{Supp}(g)$. Furthermore, since $K(\text{Supp}(f), e_B) \leq 2K(\text{Supp}(f), b)$, and similarly for (g, c) , we obtain the upper bound

$$d_B(b^k t_i, z^{-1} b^k t_i) \leq 2(d_\Gamma(1, u) + d_\Gamma(1, v))$$

using Lemma 2.1.

We finish by applying the triangle inequality and using the left-invariance of the word metric d_B as follows:

$$\begin{aligned} d_B(e_B, z^{-1}) &\leq d_B(e_B, b^k t_i) + d_B(b^k t_i, z^{-1} b^k t_i) + d_B(z^{-1} b^k t_i, z^{-1}) \\ &\leq 2d_B(e_B, b^k t_i) + 2(d_\Gamma(1, u) + d_\Gamma(1, v)) \\ &\leq 2K(\text{Supp}(f), b) + 2(d_\Gamma(1, u) + d_\Gamma(1, v)) \\ &\leq 3(d_\Gamma(1, u) + d_\Gamma(1, v)). \end{aligned}$$

This completes the proof. \square

Soon we will give Theorem 3.5, which will describe the length of short conjugators in wreath products $A \wr B$ where B is torsion-free. Before we dive into this however, it will prove useful in Section 5, when we look at conjugacy in free solvable groups, to understand how the conjugators are constructed. In particular, it is important to understand that the size of a conjugator $(h, z) \in A \wr B$ can be expressed in terms of the size of z in B with no need to refer to the function h at all. This is what we explain in Lemma 3.4.

For $b \in B$, let $\delta_{\langle b \rangle}^B(n) = \max\{m \in \mathbb{Z} \mid d_B(e_B, b^m) \leq n\}$ be the subgroup distortion of $\langle b \rangle$ in B . Fix a finite generating set X for B and let $\text{Cay}(B, X)$ be the corresponding Cayley graph.

Lemma 3.4. *Suppose $u = (f, b), v = (g, c)$ are conjugate elements in $\Gamma = A \wr B$ and let $n = d_\Gamma(1, u) + d_\Gamma(1, v)$. Suppose also that b and c are of infinite order in B . If $\gamma = (h, z)$ is a conjugator for u and v in Γ then*

$$d_\Gamma(1, \gamma) \leq (n+1)P(2\delta_{\langle b \rangle}^B(P) + 1)$$

where $P = 2d_B(1, z) + n$.

Proof. Without loss of generality we may assume $d_\Gamma(1, u) \leq d_\Gamma(1, v)$. From Lemma 3.2 we have an explicit expression for h . We use this expression to give an upper

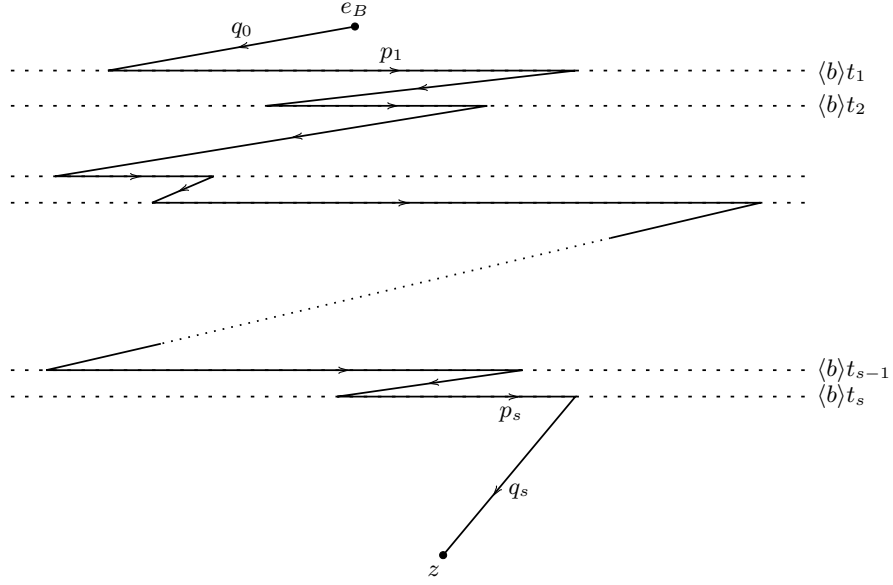


FIGURE 1. We build a path from e_B to b by piecing together paths q_i and p_i , where the paths p_i run through the intersection of $\text{Supp}(h)$ with a coset $\langle b \rangle t_i$ and the paths q_i connect these cosets.

bound for the size of (h, z) , making use of Lemma 2.1 which tells us

$$d_\Gamma(1, \gamma) = K(\text{Supp}(h), z) + |h|$$

where $K(\text{Supp}(h), z)$ is the length of the shortest path in $\text{Cay}(B, X)$ from e_B to z travelling via every point in $\text{Supp}(h)$ and $|h|$ is the sum of terms $d_A(e_A, f(x))$ over all $x \in B$.

We begin by obtaining an upper bound on the size of $K(\text{Supp}(h), z)$. To do this we build a path from e_B to z , zig-zagging along cosets of $\langle b \rangle$, see Figure 1. Lemma 3.2 tells us that there is a family of right-coset representatives $\{t_i\}_{i \in I}$ such that

$$h(b^k t_i) = \left(\prod_{j \leq k} g(z^{-1} b^j t_i) \right)^{-1} \left(\prod_{j \leq k} f(b^j t_i) \right)$$

for every $i \in I$ and $k \in \mathbb{Z}$. Notice that $b^k t_i$ is in $\text{Supp}(h)$ only if there is some $j \in \mathbb{Z}$ such that either $z^{-1} b^j t_i \in \text{Supp}(g)$ or $b^j t_i \in \text{Supp}(f)$. Let t_1, \dots, t_s be all the coset representatives for which $\text{Supp}(h)$ intersects the coset $\langle b \rangle t_i$. The number s of such cosets is bounded above by the size of the set $\text{Supp}(f) \cup \text{Supp}(g)$. Observe

$$\begin{aligned} s &\leq |\text{Supp}(f) \cup \text{Supp}(g)| \\ &\leq K(\text{Supp}(f), b) + K(\text{Supp}(g), c) \\ &\leq d_\Gamma(1, u) + d_\Gamma(1, v) = n. \end{aligned}$$

If we restrict our attention to one of these cosets, $\langle b \rangle t_i$, then there exist integers $m_1 < m_2$ such that $b^j t_i \in \text{Supp}(h)$ implies $m_1 \leq j \leq m_2$. We can choose m_1 and m_2 so that for $m \in \{m_1, m_2\}$ either $z^{-1} b^m t_i$ is in $\text{Supp}(g)$ or $b^m t_i$ is in $\text{Supp}(f)$. Let p_i be a piecewise geodesic in the Cayley graph of B which connects $b^{m_1} t_i$ to $b^{m_2} t_i$ via $b^j t_i$ for every $m_1 < j < m_2$. The length of p_i will be

$$d_B(b^j t_i, b^{j+1} t_i) \delta_{\langle b \rangle}^B(D)$$

for any $j \in \mathbb{Z}$, and where $D \leq \text{diam}(z\text{Supp}(g) \cup \text{Supp}(f))$. Choose $j \in \mathbb{Z}$ such that either $b^j t_i \in \text{Supp}(f)$ or $z^{-1}b^j t_i \in \text{Supp}(g)$. In the former case we get

$$\begin{aligned} d_B(b^j t_i, b^{j+1} t_i) &\leq d_B(b^j t_i, b) + d_B(b, b^{j+1} t_i) \\ &= d_B(b^j t_i, b) + d_B(e_B, b^j t_i) \\ &\leq K(\text{Supp}(f), b) \end{aligned}$$

where the last inequality follows because any path from e_B to b via all points in $\text{Supp}(f)$ will have to be at least as long as the path from e_B to b via the point $b^j t_i$. In the latter case, when $z^{-1}b^j t_i \in \text{Supp}(g)$, we get

$$\begin{aligned} d_B(b^j t_i, b^{j+1} t_i) &\leq d_B(b^j t_i, b) + d_B(b, b^{j+1} t_i) \\ &= d_B(z^{-1}b^j t_i, z^{-1}b) + d_B(z^{-1}, z^{-1}b^j t_i) \\ &\leq d_B(z^{-1}b^j t_i, c) + d_B(c, cz^{-1}) + d_B(z^{-1}, e_B) + d_B(e_B, z^{-1}b^j t_i) \\ &\leq P - n + K(\text{Supp}(g), c) \leq P \end{aligned}$$

where we obtain the last line because $d_B(e_B, z^{-1}b^j t_i) + d_B(z^{-1}b^j t_i, c)$ describes the length of a path in $\text{Cay}(B, X)$ from e_B to c travelling via some point in $\text{Supp}(g)$, so it must be bounded above by $K(\text{Supp}(g), c)$. Hence, in either case we get that the path p_i has length bounded above by $P\delta_{(b)}^B(D)$.

We will now show that $D \leq P$. We observed above that D is bounded above by the diameter of the set $z\text{Supp}(g) \cup \text{Supp}(f)$, which will be given by the length of a path connecting two points in it. We take a loop through e_B and all points in the set $z\text{Supp}(g) \cup \text{Supp}(f)$, a loop such as that in Figure 2. The length of this loop will certainly be bigger than the diameter. Hence we have

$$\begin{aligned} D &\leq K(z\text{Supp}(g) \cup \text{Supp}(f), e_B) \\ &\leq K(\text{Supp}(g), c) + K(\text{Supp}(f), b) + 2d_B(e_B, z) \\ &\leq P. \end{aligned}$$

For $i = 1, \dots, s-1$ let q_i be a geodesic path which connects the end of p_i with the start of p_{i+1} . Let q_0 connect e_B with the start of p_1 and q_s connect the end of p_s with z . Then the concatenation of paths $q_0, p_1, q_1, \dots, q_{s-1}, p_s, q_s$ is a path from e_B to z via every point in $\text{Supp}(h)$.

For each i , the path q_i will be a geodesic connecting two points of the set $z\text{Supp}(g) \cup \text{Supp}(f) \cup \{e_B, z\}$. The length of such a path will be bounded above

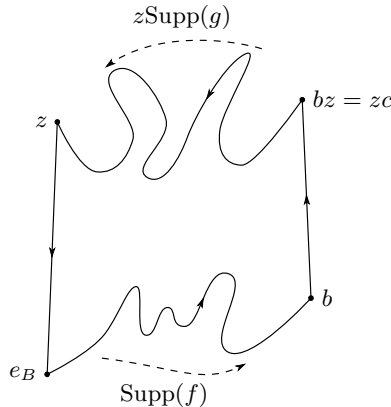


FIGURE 2. The concatenation of four paths: a path from e_B to b through $\text{Supp}(f)$, followed by a path from b to $bz = zc$, then by a path from zc to z through each point in $z\text{Supp}(g)$ and we finish off by travelling from z back to e_B .

by $K(z\text{Supp}(g) \cup \text{Supp}(f), z)$. This in turn will be bounded above by the length of the path in Figure 2, whose length we have shown is bounded above by P .

Hence our path $q_0, p_1, q_1, \dots, q_{s-1}, p_s, q_s$ has length bounded above by

$$(n+1)P + nP\delta_{\langle b \rangle}^B(P) \leq (n+1)P(\delta_{\langle b \rangle}^B(P) + 1)$$

thus giving an upper bound for $K(\text{Supp}(h), z)$.

Now we need to turn our attention to an upper bound for $|h|$. By the value of $h(b^k t_i)$ given to us by Lemma 3.2 we see that

$$\begin{aligned} d_B(e_B, h(b^k t_i)) &\leq \sum_{j \leq k} d_B(e_B, g(z^{-1} b^j t_i)) + \sum_{j \leq k} d_B(e_B, f(b^j t_i)) \\ &\leq |g| + |f| \leq n \end{aligned}$$

The number of elements $b^k t_i$ in the support of h can be counted in the following way. Firstly, the number of $i \in I$ for which $\langle b \rangle t_i \cap \text{Supp}(h) \neq \emptyset$ is equal to s , which we showed above to be bounded by n . Secondly, for each such i , recall that there exists $m_1 \leq m_2$ such that $b^j t_i \in \text{Supp}(h)$ implies $m_1 \leq j \leq m_2$. Hence for each i the number of $k \in \mathbb{Z}$ for which $b^k t_i \in \text{Supp}(h)$ is bounded above by $m_2 - m_1 \leq \delta_{\langle b \rangle}^B(P)$. So in conclusion we have

$$\begin{aligned} d_\Gamma(1, \gamma) &= K(\text{Supp}(h), z) + |h| \\ &\leq (n+1)P(\delta_{\langle b \rangle}^B(P) + 1) + n^2 \delta_{\langle b \rangle}^B(P) \\ &\leq (n+1)P(2\delta_{\langle b \rangle}^B(P) + 1) \end{aligned}$$

where $n = d_\Gamma(1, u) + d_\Gamma(1, v)$ and $P = 2d_B(e_B, z) + n$. \square

Theorem 3.5. *Suppose A is recursively presented with solvable conjugacy problem and B is torsion-free, recursively presented and has solvable conjugacy and power problems. Let $u = (f, b), v = (g, c) \in \Gamma = A \wr B$. Then u, v are conjugate if and only if there exists a conjugator $\gamma \in \Gamma$ such that*

$$d_\Gamma(1, \gamma) \leq (n+1)P(2\delta_{\langle b \rangle}^B(P) + 1)$$

where $n = d_\Gamma(1, u) + d_\Gamma(1, v)$ and $P = 7n$ if (f, b) is not conjugate to $(1, b)$ and $P = n + 2\text{CLF}_B(n)$ otherwise.

Proof. By Lemma 3.3 we can find a conjugator $\gamma = (h, z)$ which satisfies the inequality $d_B(e_B, z) \leq \text{CLF}_B(n)$ if (f, b) is conjugate to $(1, b)$ or $d_B(e_B, z) \leq 3n$ otherwise. Therefore if we set $P = n + 2\text{CLF}_B(n)$ if (f, b) is conjugate to $(1, b)$ and $P = 7n$ otherwise then the result follows immediately by applying Lemma 3.4 to the conjugator γ obtained from Lemma 3.3. \square

If we remove the condition that B is torsion-free we can still obtain some information on the conjugacy length. Theorem 3.5 does the work when we look at elements in $A \wr B$ such that the B -components are of infinite order. However, if they have finite order we need to understand the size of the conjugators α_i as in Proposition 3.1 and Lemma 3.2. When the order of b is finite, the construction of the function h by Matthews in Proposition 3.1 will work for any conjugator α_{t_i} between $\pi_{t_i}^{(z)}(g)$ and $\pi_{t_i}(f)$. Suppose a conjugacy length function exists for the group A . Then, since

$$|\pi_{t_i}^{(z)}(g)| + |\pi_{t_i}(f)| \leq |g| + |f| \leq n$$

where $n = d_\Gamma(1, u) + d_\Gamma(1, v)$, for each coset representative t_i and each $b^k \in \langle b \rangle$ we have

$$d_B(e_B, h(b^k t_i)) \leq |f| + |g| + \text{CLF}_A(n) \leq n + \text{CLF}_A(n).$$

With the aid of the conjugacy length function for A we can therefore give the following:

Lemma 3.6. *Suppose $u = (f, b), v = (g, c)$ are conjugate elements in $\Gamma = A \wr B$ and let $n = d_\Gamma(1, u) + d_\Gamma(1, v)$. Suppose also that b and c are of finite order N . If $\gamma = (h, z)$ is a conjugator for u and v in Γ then*

$$d_\Gamma(1, \gamma) \leq P(N+1)(2n + \text{CLF}_A(n) + 1)$$

where $P = 2d_B(e_B, z) + n$.

Proof. For the most part this proof is the same as for Lemma 3.4. It will differ in two places. As mentioned above, we obtain

$$d_B(e_B, h(b^k t_i)) \leq |f| + |g| + \text{CLF}_A(n)$$

for each coset representative t_i and $b^k \in \langle b \rangle$. By a similar process as that in Lemma 3.4 we deduce the upper bound

$$|h| \leq nN(n + \text{CLF}_A(n)).$$

The second place where we need to modify the proof is in the calculation of an upper bound for the length of each path p_i . Since b is of finite order, each coset will give a loop in $\text{Cay}(B, X)$. We will let p_i run around this loop, so its length will be bounded above by $Nd_B(t_i, bt_i)$. As before we get $d_B(t_i, bt_i) \leq P$, so in the upper bound obtained for $K(\text{Supp}(h), z)$ we need only replace the distortion function $\delta_{\langle b \rangle}^B$ by the order N of b in B . Thus

$$K(\text{Supp}(h), z) \leq P(N+1)(n+1)$$

where $P = 2d_B(e_B, z) + n$. Combining this with the upper bound above for $|h|$ we get

$$\begin{aligned} d_\Gamma(1, \gamma) &\leq P(N+1)(n+1) + nN(n + \text{CLF}_A(n)) \\ &\leq P(N+1)(2n + \text{CLF}_A(n) + 1). \end{aligned}$$

□

We finish this section by applying Lemma 3.3 and Lemma 3.6 to give the complete picture for the length of short conjugators in the case when B may contain torsion.

Theorem 3.7. *Suppose A is recursively presented with solvable conjugacy problem and B is recursively presented and has solvable conjugacy and power problems. Let $u = (f, b), v = (g, c) \in \Gamma$ where the order of b and c is $N \in \mathbb{N} \cup \{\infty\}$. Then u, v are conjugate if and only if there exists a conjugator $\gamma \in \Gamma$ such that either*

$$\begin{aligned} d_\Gamma(1, \gamma) &\leq P(N+1)(2n + \text{CLF}_A(n) + 1) && \text{if } N \text{ is finite; or} \\ d_\Gamma(1, \gamma) &\leq (n+1)P(2\delta_{\langle b \rangle}^B(P) + 1) && \text{if } N = \infty, \end{aligned}$$

where $n = d_\Gamma(1, u) + d_\Gamma(1, v)$ and $P = 7n$ if (f, b) is conjugate to $(1, b)$ or $P = n + 2\text{CLF}_B(n)$ otherwise.

4. SUBGROUP DISTORTION

We saw in Theorem 3.5 that in order to understand the conjugacy length function of a wreath product $A \wr B$ we need to understand the distortion function for infinite cyclic subgroups in B .

Recall subgroup distortion is studied up to an equivalence relation of functions. For functions $f, g : \mathbb{N} \rightarrow [0, \infty)$ we write $f \preceq g$ if there exists an integer $C > 0$ such that $f(n) \leq Cg(Cn)$ for all $n \in \mathbb{N}$. The two functions are equivalent if both $f \preceq g$ and $g \preceq f$. In this case we write $f \asymp g$.

We will see that all cyclic subgroups of free solvable groups $S_{r,d}$ are undistorted. This is not always the case in finitely generated solvable groups. For example, in the solvable Baumslag-Solitar groups $BS(1, q) = \langle a, b \mid aba^{-1} = b^q \rangle$ the subgroup generated by b is at least exponentially distorted since $b^{q^n} = a^n b a^{-n}$. Because this

type of construction doesn't work in $\mathbb{Z} \wr \mathbb{Z}$ or free metabelian groups it lead to a question of whether all subgroups of these groups are undistorted (see [DO11, §2.1]). Davis and Olshanskii answered this question in the negative, giving, for any positive integer t , 2-generated subgroups of these groups with distortion function bounded below by a polynomial of degree t .

The following Lemma is given in [DO11, Lemma 2.3].

Lemma 4.1. *Let A, B be finitely generated abelian groups. Then every finitely generated abelian subgroup of $A \wr B$ is undistorted.*

Davis and Olshanskii prove this by showing that such subgroups are retracts of a finite index subgroup of $A \wr B$. A similar process can be applied to finitely generated abelian subgroups of free solvable groups to show that they are undistorted [Ols]. Below we give an alternative proof for cyclic subgroups which provides an effective estimate for the constant and which uses only results given in this paper.

Proposition 4.2. *Every cyclic subgroup of a free solvable group is undistorted. In particular, suppose $d \geq 1$ and let x be a non-trivial element of $S_{r,d}$. Then*

$$\delta_{\langle x \rangle}^{S_{r,d}}(n) \leq 2n.$$

Proof. Let w be a non-trivial element of the free group F . There exists an integer c such that $w \in F^{(c)} \setminus F^{(c+1)}$, where we include the case $F^{(0)} = F$. First we suppose that $d = c + 1$.

If $c = 0$ then we have $x \in F/F' = \mathbb{Z}^r$ and we apply linear distortion in \mathbb{Z}^r . If $c > 0$ then we take a Magnus embedding $\varphi : S_{r,d} \hookrightarrow \mathbb{Z}^r \wr S_{r,c}$ and observe that since $w \in F^{(c)} \setminus F^{(d)}$ the image of x in φ is $(f, 1)$ for some non-trivial function $f : S_{r,c} \rightarrow \mathbb{Z}^r$. If f^k denotes the function such that $f^k(b) = kf(b)$ for $b \in S_{r,c}$, then for any $k \in \mathbb{Z}$, since the Magnus embedding is 2-bi-Lipschitz (Theorem 2.6),

$$d_{S_{r,d}}(1, x^k) \geq \frac{1}{2} d_M(1, (f, 1)^k) = \frac{1}{2} d_M(1, (f^k, 1)).$$

We can apply Lemma 2.1 to get

$$\frac{1}{2} d_M(1, (f^k, 1)) = \frac{1}{2} \left(K(\text{Supp}(f), 1) + \sum_{b \in S_{r,c}} |kf(b)| \right)$$

and since the image of f lies in \mathbb{Z}^r and f is non-trivial

$$\sum_{b \in S_{r,c}} |kf(b)| = |k| \sum_{b \in S_{r,c}} |f(b)| \geq |k|.$$

Hence

$$d_{S_{r,d}}(1, x^k) \geq \frac{1}{2} d_M(1, (f^k, 1)) \geq \frac{1}{2} |k|.$$

This implies $\delta_{\langle x \rangle}^{S_{r,d}}(n) \leq 2n$.

Now suppose that $d > c + 1$. Then we define a homomorphism

$$\psi : S_{r,d} \rightarrow S_{r,c+1}$$

by sending the free generators of $S_{r,d}$ to the corresponding free generator of $S_{r,c+1}$. Then, as before, set $x = wF^{(d)}$ and define y to be the image of x under ψ . By the construction of ψ we have that $y = wF^{(c+1)}$. Note that y is non-trivial and ψ does not increase the word length. Hence, using the result above for $S_{r,c+1}$, we observe that

$$d_{S_{r,d}}(1, x^k) \geq d_{S_{r,c+1}}(1, y^k) \geq \frac{1}{2} |k|.$$

This suffices to show that the distortion function is always bounded above by $2n$. \square

5. CONJUGACY IN FREE SOLVABLE GROUPS

The following Theorem was given by Remeslennikov and Sokolov [RS70]. It allowed them to prove the solubility of the conjugacy problem in free solvable groups.

Theorem 5.1 (Remeslennikov–Sokolov [RS70]). *Suppose F/N is torsion-free and let $u, v \in F/N'$. Then $\varphi(u)$ is conjugate to $\varphi(v)$ in $M(F/N)$ if and only if u is conjugate to v in F/N' .*

The proof of Theorem 5.2 more-or-less follows Remeslennikov and Sokolov's proof of the preceding theorem. With it one can better understand the nature of conjugators and how they relate to the Magnus embedding. In particular it tells us that once we have found a conjugator in the wreath product $\mathbb{Z}^r \wr S_{r,d-1}$ for the image of two elements in $S_{r,d}$, we need only modify the function component of the element to make it lie in the image of the Magnus embedding.

Theorem 5.2. *Let u, v be two elements in F/N' such that $\varphi(u)$ is conjugate to $\varphi(v)$ in $M(F/N)$. Let $g \in M(F/N)$ be identified with $(f, \gamma) \in \mathbb{Z}^r \wr F/N$. Suppose that $\varphi(u)g = g\varphi(v)$. Then there exists $w \in F/N'$ such that $\varphi(w) = (f_w, \gamma)$ is a conjugator.*

Proof. Let $g \in M(F/N)$ be such that $\varphi(u)g = g\varphi(v)$. Suppose

$$g = \begin{pmatrix} \gamma & a \\ 0 & 1 \end{pmatrix} = (f, \gamma)$$

for $\gamma \in F/N$ and $a \in \mathcal{R}$. Recall that $\alpha : F/N' \rightarrow F/N$ is the canonical homomorphism and by Lemma 2.3 we may consider the derivations $\frac{\partial^*}{\partial x_i}$ to be maps from $\mathbb{Z}(F/N')$ rather than $\mathbb{Z}(F)$. By direct calculation we obtain the two equations

$$(3) \quad \alpha(u)\gamma = \gamma\alpha(v)$$

$$(4) \quad \sum_{i=1}^r \frac{\partial^* u}{\partial x_i} t_i + \alpha(u)a = \gamma \sum_{i=1}^r \frac{\partial^* v}{\partial x_i} t_i + a$$

We now split the proof into two cases, depending on whether or not $\alpha(u)$ is the identity element.

CASE 1: $\alpha(u)$ is trivial

In this case equation (4) reduces to

$$\sum_{i=1}^r \frac{\partial^* u}{\partial x_i} t_i = \gamma \sum_{i=1}^r \frac{\partial^* v}{\partial x_i} t_i$$

and it follows that $\varphi(\gamma_0) = (h, \gamma)$ will be a conjugator for $\varphi(u)$ and $\varphi(v)$, where γ_0 is any lift of γ in F/N' .

CASE 2: $\alpha(u)$ is non-trivial

Note that in this case we actually show a stronger result, that any conjugator for $\varphi(u)$ and $\varphi(v)$ must lie in the subgroup $\varphi(F/N')$. This is clearly not necessarily true in the first case.

First conjugate $\varphi(u)$ by $\varphi(\gamma_0)$, where γ_0 is a lift of γ in F/N' . This gives us two elements which are conjugate by a unipotent matrix in $M(F/N)$, in particular there exist b_1, \dots, b_r in $\mathbb{Z}(F/N)$ such that the conjugator is of the form

$$\varphi(\gamma_0)^{-1}g = \gamma' = \begin{pmatrix} 1 & b_1 t_1 + \dots + b_r t_r \\ 0 & 1 \end{pmatrix}.$$

Hence the aim now is to show that there is some element y in N such that $\gamma' = \varphi(y)$, in particular $\frac{\partial^* y}{\partial x_i} = b_i$ for each i . Therefore, without loss of generality, we assume that $\varphi(u)$ and $\varphi(v)$ are conjugate by such a unipotent matrix.

Assume that $\varphi(u)\gamma' = \gamma'\varphi(v)$. Then equation (3) tells us that $\alpha(u) = \alpha(v)$. Hence $uv^{-1} = z \in N$. Observe that $\frac{\partial^* z}{\partial x_i} = \frac{\partial^* u}{\partial x_i} - \frac{\partial^* v}{\partial x_i}$, hence from equation (4) we get

$$(1 - \alpha(u))b_i = \frac{\partial^* z}{\partial x_i}$$

for each $i = 1, \dots, r$.

Let c be an element of $\mathbb{Z}(F/N')$ such that $\frac{\partial^* c}{\partial x_i} = b_i$ for each i . We therefore have the following:

$$(1 - \alpha(u)) \sum_{i=1}^r \frac{\partial^* c}{\partial x_i} (\alpha(x_i) - 1) = \sum_{i=1}^r \frac{\partial^* z}{\partial x_i} (\alpha(x_i) - 1).$$

We can choose c so that $\varepsilon(c) = 0$, and then apply the fundamental formula of Fox calculus, Lemma 2.2, to both sides of this equation to get

$$(1 - \alpha(u))c = z - 1$$

since $z \in N$ implies $\varepsilon(z) = 1$. In $\mathbb{Z}(F/N)$ the right-hand side is 0. Furthermore, since F/N is torsion-free, $(1 - \alpha(u))$ is not a zero divisor, so c lies in the kernel of the homomorphism $\alpha^* : \mathbb{Z}(F/N') \rightarrow \mathbb{Z}(F/N)$. Hence there is an expression of c in the following way (see Lemma 2.4):

$$c = \sum_{j=1}^m r_j (h_j - 1)$$

where m is a positive integer and for each $j = 1, \dots, m$ we have $r_j \in F/N'$ and $h_j \in N/N'$. Differentiating this expression therefore gives

$$\begin{aligned} b_i = \frac{\partial^* c}{\partial x_i} &= \sum_{j=1}^m \left(\frac{\partial^* r_j}{\partial x_i} \varepsilon(h_j - 1) + \alpha(r_j) \frac{\partial^* (h_j - 1)}{\partial x_i} \right) \\ &= \sum_{j=1}^m \alpha(r_j) \frac{\partial^* h_j}{\partial x_i} \end{aligned}$$

where α is the quotient map $\alpha : F/N' \rightarrow F/N$. We set

$$y = \prod_{j=1}^m r_j h_j r_j^{-1} \in N.$$

Since $h_j \in N/N'$ for each j , we have the following equations:

$$\frac{\partial^* (h_1 h_2)}{\partial x_i} = \frac{\partial^* h_1}{\partial x_i} + \frac{\partial^* h_2}{\partial x_i}, \quad \frac{\partial^* (r_j h_j r_j^{-1})}{\partial x_i} = r_j \frac{\partial^* h_j}{\partial x_i}.$$

Using these, the condition $\frac{\partial^* y}{\partial x_i} = b_i$ can be verified. Hence, $g = \varphi(\gamma_0)\varphi(y)$, so taking $w = \gamma_0 y$ gives us a conjugator $\varphi(\gamma_0 y) = (f_w, \gamma)$ of the required form. \square

We now use Theorem 5.2 and work from Section 3 to give an estimate of the conjugacy length function of a group F/N' with respect to the conjugacy length function of F/N and the subgroup distortion of its cyclic subgroups. Recall that $\alpha : F/N' \rightarrow F/N$ denotes the canonical homomorphism and $\delta_{\langle \alpha(u) \rangle}^{F/N}$ is the distortion function for the subgroup of F/N generated by $\alpha(u)$. In particular observe that if $\text{CLF}_{F/N}$ and $\delta_{\langle \alpha(u) \rangle}^{F/N}$ are bounded by polynomial functions then $\text{CLF}_{F/N'}$ is also bounded by a polynomial.

Theorem 5.3. *Let N be a normal subgroup of F such that F/N is torsion-free and has solvable conjugacy and power problems. Let u, v be elements in F/N' . Then u, v are conjugate in F/N' if and only if there exists a conjugator $\gamma \in F/N'$ such that*

$$d_{F/N'}(1, \gamma) \leq 2(2n + 1)P(2\delta_{\langle \alpha(u) \rangle}^{F/N}(P) + 1)$$

where $n = d_{F/N'}(1, u) + d_{F/N'}(1, v)$ and $P = 14n$ if $\varphi(u)$ is not conjugate to $(1, \alpha(u))$ or $P = 2n + 2\text{CLF}_{F/N}(2n)$ otherwise.

Proof. We first choose a conjugator $(h, \gamma) \in \mathbb{Z}^r \wr F/N$ for which γ is short, as in Lemma 3.3. Then Theorem 5.2 tells us that there exists some lift γ_0 of γ in F/N' such that $\varphi(\gamma_0) = (h_0, \gamma)$ is a conjugator. Lemma 3.4 applies in this situation to give us

$$d_M(1, \varphi(\gamma_0)) \leq (n' + 1)P'(2\delta_{\langle \alpha(u) \rangle}^{F/N}(P') + 1)$$

where $n' = d_M(1, \varphi(u)) + d_M(1, \varphi(v))$ and $P' = 7n'$ if $\varphi(u)$ is not conjugate to $(1, \alpha(u))$ or $P' = n' + 2\text{CLF}_{F/N}(n')$ otherwise. By Theorem 2.6 we see that $d_{F/N'}(1, \gamma) \leq 2d_M(1, \gamma_0)$ and $n' \leq 2n$. Hence the result follows. \square

In the special case where $N' = F^{(d)}$ the quotient F/N' is the free solvable group $S_{r,d}$ of rank r and derived length d . Plugging Proposition 4.2 into Theorem 5.3 gives us an upper bound for the length of short conjugators between two elements in free solvable groups.

Corollary 5.4. *Let $d \geq 1$ and $r > 1$. Then the conjugacy length function of $S_{r,d}$ is a polynomial of degree at most $2^{d+1} - 1$.*

Proof. In light of Proposition 4.2, applying Theorem 5.3 gives us a conjugator $\gamma \in S_{r,d}$ such that

$$d_{S_{r,d}}(1, \gamma) \leq 2(2n + 1)P(2P + 1).$$

where either $P = 14n$ or $P = 2n + 2\text{CLF}_{S_{r,d-1}}(2n)$. We proceed by induction on derived length d . The base case is $S_{r,1} = \mathbb{Z}^r \wr \mathbb{Z}^r$. Applying Theorem 3.5 tells us that we can find a conjugator of length bounded above by

$$(n + 1)7n(14n + 1)$$

since the conjugacy length function for an abelian group is the zero function. Furthermore, from Theorem 5.3, we observe that, if the conjugacy length function of $S_{r,d-1}$ is bounded above by a polynomial of degree m , then the conjugacy length function of $S_{r,d}$ will be bounded above by a polynomial of $2m + 1$. This gives us a sequence (p_d) defined iteratively as $p_1 = 3$ and $p_{d+1} = 2p_d + 1$. The solution to this is $p_d = 2^{d+1} - 1$, as required. \square

6. COMPRESSION EXPONENTS

We can use the fact that the Magnus embedding is a quasi-isometric embedding to obtain a lower bound for the L_p compression exponent of free solvable groups. The L_p compression exponent is a way of measuring how a group embeds into L_p . A non-zero L_p compression exponent implies the existence of a uniform embedding of G into L_p , a notion which was introduced by Gromov [Gro93]. Gromov claimed that if a group admits a uniform embedding into a Hilbert space then the Novikov conjecture holds in this group. The claim was later proved by Yu [Yu00], where he also showed that amenable groups admit such embeddings. Kasparov and Yu [KY06] extended this result to groups which admit a uniform embedding into any uniformly convex Banach space.

Let G be a finitely generated group with word metric denoted by d_G and let Y be a metric space with metric d_Y . A map $f : G \rightarrow Y$ is called a *uniform embedding* if there are two functions $\rho_{\pm} : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ such that $\rho_{-}(r) \rightarrow \infty$ as $r \rightarrow \infty$ and

$$\rho_{-}(d_G(g_1, g_2)) \leq d_Y(f(g_1), f(g_2)) \leq \rho_{+}(d_G(g_1, g_2))$$

for $g_1, g_2 \in G$.

One can define the L_p *compression exponent* for a finitely generated group G , denoted by $\alpha_p^*(G)$, to be the supremum over all $\alpha \geq 0$ such that there exists a Lipschitz map $f : G \rightarrow L_p$ satisfying

$$C d_G(g_1, g_2)^\alpha \leq \|f(g_1) - f(g_2)\|$$

for any positive constant C . For $p = 2$, the Hilbert compression exponent, which is denoted by $\alpha^*(G)$, was introduced by Guentner and Kaminker [GK04].

When we restrict to G -equivariant embeddings $f : G \rightarrow L_p$ and take the same supremum we obtain the *equivariant L_p compression exponent* for G , denoted by $\alpha_p^\#(G)$. In the case when G is amenable, the Hilbert compression exponent is equal to the equivariant Hilbert compression exponent - this was first proved for abelian groups by Aharoni, Maurey and Mityagin [AMM85] and in general by Gromov, with a full proof given by de Cornulier, Tessera and Valette [dCTV07].

Of particular interest to us is what happens to compression under taking a wreath product. Naor and Peres give a lower bound for the compression of $A \wr B$ when B is of polynomial growth [NP11, Theorem 3.1].

Theorem 6.1 (Naor–Peres [NP11]). *Let A, B be finitely generated groups such that B has polynomial growth. Then, for $p \in [1, 2]$,*

$$\alpha_p^*(A \wr B) \geq \min \left\{ \frac{1}{p}, \alpha_p^*(A) \right\}.$$

Li showed in particular that a positive compression exponent is preserved by taking wreath products [Li10].

Theorem 6.2 (Li [Li10]). *Let A, B be finitely generated groups. For $p \geq 1$ we have*

$$\alpha_p^*(A \wr B) \geq \max \left\{ \frac{1}{p}, \frac{1}{2} \right\} \min \left\{ \alpha_1^*(A), \frac{\alpha_1^*(B)}{1 + \alpha_1^*(B)} \right\}.$$

We can deduce from the result of Naor and Peres that the L_1 compression exponent for $\mathbb{Z}^r \wr \mathbb{Z}^r$ is equal to 1. Hence the L_1 compression exponent for metabelian groups, using Theorem 2.6, is also equal to 1. Then by induction on the derived length, first for $p = 1$ then for all $p > 1$, we can use Li's result to obtain the following lower bounds for all free solvable groups:

Corollary 6.3. *Let $r, d \in \mathbb{N}$. Then*

$$\alpha_1^*(S_{r,d}) \geq \frac{1}{d-1}$$

and for $p > 1$

$$\alpha_p^*(S_{r,d}) \geq \frac{1}{d-1} \min \left\{ \frac{1}{p}, \frac{1}{2} \right\}.$$

It would be interesting to determine whether $\alpha_p^*(S_{r,d})$ is strictly less than $\frac{1}{2}$ since no solvable group is known to have non-zero Hilbert compression exponent strictly less than $\frac{1}{2}$, although Austin [Aus11] has shown that there exist solvable groups with L_p compression exponent equal to zero.

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